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CONSTRUCTION OF SOLITARY WAVE SOLUTIONS OF THE KORTEWEG-DE-VRIES EQUATION VIA PAINLEVE ANALYSIS

W. Hereman[1,*], P. P. Banerjee[1,**] and D. Faker[2]

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ABSTRACT

Starting from the Painleve-Backlund equations obtained from Painleve analysis of the Korteweg-de Vries equation, closed form solitary wave solutions are explicitly constructed. It is shown that repetitive application of the Mobius group of fractional linear transformations does not lead to new solutions. Various connections of the Painleve method with Hirota's formalism, the Backlund transformation method, the Lax approach and the Inverse Scattering Technique are discussed in detail.

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KEY WORDS: soliton theory, KdV, Painleve analysis, solitary wave solutions, direct methods.

partial differential equations (KP)

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CONSTRUCTION OF SOLITARY WAVE SOLUTIONS OF THE KORTEWEG-DE-VRIES EQUATION VIA PAINLEVE ANALYSIS

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1. Introduction

Over the last five years, Painleve analysis of ordinary (ODEs) and partial differential equations (PDEs) has been successfully applied to get an idea whether or not the given equation or system of equations might be integrable. In essence the Painleve test verifies if the solutions of the differential equations are free of movable critical points; in other words, the only movable (i.e., dependent on the initial conditions) singularities should be poles.

With reference to systems of nonlinear coupled ODEs, the Painleve test provides insight into the possibility of constructing invariants (first integrals) and their nature. Direct application of the Painleve test at the PDE level provides an algorithm to construct the infinite set of conserved quantities. Furthermore, Painleve analysis serves as an elegant tool to construct the Lax pair, and thus allows to linearize and subsequently solve the PDE by the Inverse Scattering Transform (IST). Painleve analysis also leads to non-standard auto-Backlund transformations by suitable truncation of the Laurent expansion of the solution. For ample references on those aspects the reader should consult the papers of Weiss (1983-1987).

In this paper we will focus on yet another spin-off of the Painleve analysis of PDEs. More precisely, we will investigate if the Painleve test helps to construct a hierarchy of solitary wave solutions to the Korteweg-de Vries (KdV) equation, which is the ubiquitous equation in soliton theory. Indeed, Painleve analysis leads in a natural way to the classical (regular and singular) solitary wave solutions. Unfortunately, the Painleve approach does not permit to go beyond the known closed form traveling wave solutions to the KdV. Although the associated Painleve-Backlund (PB) equations are invariant under two distinct classes of Schlesinger transformations, the application of the latter does not lead to new results. For instance, the Mobius transformation maps solutions of the PB equations to solutions to the same equations but with different values of the parameters. Within the Mobius group we only find closed form solutions to the KdV which are disguises of the well-known sech^2 and cosech^2 solutions. Application of the Schlesinger transformations to rational solutions is much more fruitful. For the KdV and many other equations an infinite family of rational solutions can be constructed in this way. This topic, which is extensively discussed and referenced by Weiss (1983-1987) is outside the scope of this paper.

The paper is organized as follows: In Section 2, we briefly review the basics of Painleve analysis and derive the PB equations for the KdV, which serves as our paradigm throughout the paper. We then discuss in quite some detail the properties of the invariant quantities (such as the Schwarzian derivative) occuring in the PB equations. The most important result is that these quantities themselves either satisfy the KdV or the modified Korteweg-de Vries (MKdV) equation.

Next, in Section 3, starting from the PB equations, we construct the explicit (closed form) traveling wave solutions to the KdV. The validity and also the limitations of this technique are rigorously proved in a couple of theorems. The method is illustrated in an example.

In Section 4, we turn to a study of the various connections of the Painleve approach with other well established methods. We start with Hirota's formalism, which leads to N solitary wave solutions whereas the Painleve technique apparently does not. Second, we compare with the Lax method which allows us to linearize the KdV. It is shown how the Schrodinger equation and the evolution equation for the eigenfunctions follow from the PB equations by the balance of nonlinearity and dispersion (or dissipation) in the given PDE. For particular values of j, which are called the resonances and are denoted by r, it will be possible to take $u_r(x,t)$ arbitrary provided the remaining terms in the recursion relation vanish (compatibility condition). More details on the Painleve test, including the results for various PDEs, may be found in papers by Weiss (1983-1987), Gibbon et al (1985), and Hereman and Van den Bulck (1988). The latter contains a MACSYMA program that symbolically performs all the required steps for the P-test for a given ODE or PDE.

Whether or not the Painleve test allows us to draw strong conclusions about the integrability (in the context of IST) of a given nonlinear PDE remains debatable. As a matter of fact there are counterexamples available (Gibbon et al 1985, Clarkson 1985, 1986, Fokas 1987) disproving the necessity and sufficiency of the Painleve property to guarantee complete integrability of PDEs. We stress that integrability, by no means, refers to the possibility of constructing particular analytical solutions of PDEs, which we will explore in this paper. To say the least, Painleve analysis serves as a handy tool to investigate the existence of solitary wave solutions of nonlinear evolution and wave equations, which may then be constructed by any existing direct method (for a review see Hereman et al 1986). More interesting to us is the derivation of auto-Backlund transformations via a truncation of (1) and the subsequent construction of the Lax pair (Lax 1968) for the PDE. A truncation of the Laurent series is possible provided the coefficient of the constant level term is a solution to the original equation. Furthermore, the function ϕ must satisfy a set of PB equations. These equations are quite easy to solve, specially since they are transformable into an equivalent set which is invariant under the (Mobius) group of homographic (Schlesinger) transformations.

as well. Finally, since a non-standard auto-Backlund transformation is obtained by truncation of the Laurent expansion, a comparison with the classical Backlund transformation method seemed in place.

The methods presented in this paper carry over to a large class of completely integrable evolution and wave equations. As a matter of fact, a study of the Kuramoto-Shivashinsky (KS) equation along the same line of thought is presented in a paper by Conte and Musette (1988a) in volume I of these Proceedings.

2. Painleve Analysis

Weiss et al (1983) proposed a direct procedure to test whether or not a partial differential equation is of the Painleve type (P-type), i.e., its solutions admit only poles as movable singularities. The test requires the substitution of a Laurent expansion for the solution of the PDE in terms of a new dependent variable ϕ, say

$$u(x,t) = \phi^N(x,t) \sum_{j=0}^{\infty} u_j(x,t)\phi^j(x,t), \tag{1}$$

and subsequent verification if the recursion relation for the coefficients $u_j(x,t)$ has self-consistent solutions in terms of the time and spatial derivatives of ϕ. We require $\phi(x,t)$ to be analytic in some neighborhood of the singular manifold ($\phi(x,t)=0$), which itself is supposed to be non-characteristic ($\phi_x \phi_t \neq 0$). The exponent N of the singularity is determined

Consider the KdV equation in $u(x,t)$:

$$u_t + \alpha u u_x + \delta u_{3x} = 0, \tag{2}$$

where the (rescalable) real coefficients α and δ are left in for easy comparison with the literature. By now it is well-known (Weiss et al 1983, Weiss 1983, 1984a) that (2) passes the P-test and that the expansion (1) (with $N = -2$) can be truncated at the constant level term u_2, to give

$$u(x,t) = -\frac{12\delta}{\alpha}\left(\frac{\phi_x^2}{\phi^2} - \frac{\phi_{2x}}{\phi}\right) + u_2(x,t)$$

$$= \frac{12\delta}{\alpha}(\ell n\ \phi)_{2x} + u_2(x,t), \tag{3}$$

provided u_2 itself is a solution of (2), and ϕ satisfies two PB equations simultaneously. An easy way to rederive these equations is as follows. Subtract the KdV in u_2 from (2); this gives

$$(u-u_2)_t + \frac{\alpha}{2}[(u-u_2)^2 + 2u_2(u-u_2)]_x + \delta(u-u_2)_{3x} = 0 \tag{4}$$

Next, substitute $u-u_2$ from (3) into (4) and integrate once with respect to x. Setting the coefficients in ϕ^{-1} and ϕ^{-2} equal to zero results in

$$\phi_{xt} + \alpha u_2\phi_{2x} + \delta\phi_{4x} = 0, \tag{5}$$

$$\phi_x\phi_t + \alpha u_2\phi_x^2 + 4\delta\phi_x\phi_{3x} - 3\delta\phi_{2x}^2 = 0. \tag{6}$$

The coefficients of ϕ^{-3} and ϕ^{-4} vanish identically.

Upon elimination of u_2 between (5) and (6) and subsequent integration with respect to x, one obtains

$$C + \delta S = \lambda(t) \tag{7}$$

where the integration constant λ may still depend on t, and where (Conte and Musette 1988a)

$$C = \frac{\phi_t}{\phi_x} \tag{8}$$

and where

$$S = \left\{\phi;x\right\} = \frac{\phi_{3x}}{\phi_x} - \frac{3}{2}\left(\frac{\phi_{2x}}{\phi_x}\right)^2 \tag{9}$$

denotes the Schwarzian derivative (Hille 1976). Apart from the basic ratios C and S, which, as is easy to show, are both invariant under the (Mobius) group of real fractional linear transformations,

$$\phi \rightarrow \frac{a + b\phi}{c + d\phi}, \quad ad-bc \neq 0, \tag{10}$$

there is a third ratio

$$E = \frac{\phi_{2x}}{\phi_x}, \tag{11}$$

which is not invariant under (10), but nevertheless plays a crucial role, as we will see later on. Indeed, from (6), u_2 may be expressed as

$$u_2 = -\frac{1}{\alpha}(C + 4\delta S + 3\delta E^2). \tag{12}$$

Using (8) and (9), one first calculates

$$E_x = S + \frac{1}{2}E^2, \tag{13}$$

$$E_t = C_{2x} + EC_x + CS + \frac{1}{2}E^2C, \tag{14}$$

and later on expresses the relevant derivatives of u_2 in terms of C, S, their derivatives and E. Straightforward algebra reveals

$$(u_2)_t = -\frac{1}{\alpha}(C_t + 4SS_t - 6\delta^2 ES_{2x} - 6\delta^2 ES_x$$
$$+ 6\delta ESC + 3\delta E^3 C), \tag{15}$$

$$(u_2)_x = -\frac{3\delta}{\alpha}(S_x + 2ES^2 + E^3) \;, \tag{16}$$

$$(u_2)_{3x} = -\frac{3\delta}{\alpha}(S_{3x} + 6SS_x + 8ES^2 + 10E^3S$$

$$+ 5E^2S_x + 2ES_{2x} + 3E^5), \tag{17}$$

where we have used (7) to eliminate the spatial derivatives of C in favor of those of S.

Combination of (15), (16) and (17) according the KdV equation (2) gives, after simplification,

$$C_t + 5\delta S - 6\delta SC_x - 5\delta CS_x - CC_x + 4\delta^2 S_{3x} + 2\delta^2 SS_x = 0 \tag{18}$$

or equivalently with (7),

$$S_t - \lambda S_x + 3\delta SS_x + \delta S_{3x} = -\frac{\lambda_t}{4\delta} \,. \tag{19}$$

Since C and S are dependent through ϕ, the compatibility condition $\phi_{t,3x} = \phi_{3x,t}$ implies the constraint

$$S_t - C_{3x} - 2SC_x - CS_x = 0. \tag{20}$$

Using (7) once again, one thus obtains

$$S_t - \lambda S_x + 3\delta SS_x + \delta S_{3x} = 0 \tag{21}$$

KdV equation. This remarkable result is due to Weiss (1984a, 1986b) and was rediscovered by Conte and Musette (1988b), who brought it to our attention. Since (20) was derived independently of (5), (6) and (7) it is clear from comparison of (19) and (21) that $\lambda_t = 0$. This observation allows us to conclude that for constant λ, the requirement that u_2 satisfies the KdV is equivalent to a compatibility condition in terms of the function ϕ. After all, this should not surprise us since our non-standard derivation of the Painleve-

Backlund equations (5) and (6) was contingent upon u_2 solving (2). It now suffices to solve (7), with constant λ. Even this can be slightly simplified by realizing that (7) is Galilean invariant: If $\phi(x,t;\lambda)$ denotes the solution for arbitrary λ, then $\phi(x,t;0) = \phi(x-\lambda t,t;\lambda)$. Next, (5) needs to be solved for u_2. Then, substitution of ϕ and u_2 into (3) leads, at least in principle, to a new solution u to the KdV. At this point, one might wonder if C and E satify any particular well-known equations. From (7) it follows that

$$C_t + 2\lambda S_x - 3CC_x + \delta C_{3x} = 0. \tag{22}$$

The equation for E can be drived as follows. From (7) we calculate

$$C_{2x} = E_t - (EC)_x \tag{23}$$

using the compatibility condition $\phi_{t,2x} = \phi_{2x,t}$. Since C can be expressed in terms of S by (7), and S relates to E by (13), we obtain from (23),

$$E_t - \lambda E_x - \frac{3}{2}\delta E^2 E_x + \delta E_{3x} = 0, \tag{24}$$

which is the modified KdV (mKdV), a result also due to Weiss (1986b).

3. Solitary Wave Solutions Constructed from the Painleve-Backlund Equations

In our search for closed form solutions to the KdV equations we adhere to the following strategy:

(i) Starting from a traveling wave solution ϕ in real exponential form, we calculate the dispersion law from (7); λ enters a free constant parameter;

(ii) Next, we obtain u_2 from (5);

(iii) Substituting ϕ and u_2 into (3), we find u;

(iv) Subsequently applying the Mobius transformation (10) we obtain a new ϕ and repeat the above steps.

$$\frac{b}{d} + \frac{(ad-bc)}{d}\,\tilde{\psi} = \frac{a+b\phi}{c+d\phi} .$$

<u>Theorem 2</u>

Starting from

$$\phi = \exp(kx - \omega t + \theta_0) = e^{\theta} \tag{28}$$

with θ_0 constant, application of the Mobius transformation

$$\phi \rightarrow \psi = \frac{a+b\phi}{c+d\phi} \tag{29}$$

will not lead to a more general solution to the KdV than the one obtained from the special case $c = 1$, $d = 0$, a and b arbitrary.

<u>Proof</u>

Since C and S are invariant under the Mobius group (29) one can evaluate these for ϕ as in (28). With (7) we have

$$C = -\frac{\omega}{k} = \lambda + \frac{1}{2}\,\delta k^2 , \tag{30}$$

$$S = -\frac{1}{2}k^2 . \tag{31}$$

As seen from (12), only E needs to be evaluated to find u_2. Under the Mobius transformation

$$E = k\,\frac{(c-de^{\theta})}{(c+de^{\theta})} \tag{32}$$

which, remarkably enough, no longer depends on a and b. From (12), obviously

$$u_2 = -\frac{1}{\alpha}\,(\lambda + \frac{3}{2}\,\delta k^2) + \frac{12\delta}{\alpha}\,[\ell n(c+de^{\theta})]_{2x} . \tag{33}$$

Analogously, from (3),

$$u = -\frac{1}{\alpha}\,(\lambda + \frac{3}{2}\,\delta k^2) + \frac{12\delta}{\alpha}\,[\ell n(a+be^{\theta})]_{2x}. \tag{34}$$

This procedure, although beautiful in theory, is of limited practical use, as will be shown in the next theorems and the subsequent example.

Theorem 1

Suppose u and u_2 are solutions to the KdV equation (2) for which the difference is expressible in terms of the second logarithmic derivative of a singular manifold ϕ, which is linked to u_2 by the PB equations (5) and (6). Then $\psi = \frac{a + b\phi}{c + d\phi}$ will satisfy the PB equations (5) and (6) provided we replace u_2 by u.

Proof

We will first show the result for the special case

$$\psi = \frac{1}{\phi} . \tag{25}$$

Regarding the desired result, suppose that ψ satisfies

$$\psi_{xt} + \alpha v_2 \psi_{2x} + \delta \psi_{4x} = 0. \tag{26}$$

From (25) we convert all the needed derivatives of ψ into derivatives of ϕ, and we substitute into (26). Using (5) and (6) we eliminate the t derivatives of ϕ, elegantly introducing u_2 into the expression. Finally, we calculate v_2 as

$$v_2 = u_2 + \frac{12\delta}{\alpha} \left(\frac{\phi_{2x}\phi - \phi_x^2}{\phi^2} \right) = u. \tag{27}$$

Remark that (5) and (6) are invariant for scaling of ϕ and addition of a constant to it ($\phi \to c + d\phi$), likewise (26) is invariant upon replacing ψ by $\tilde{a} + \tilde{b}\psi$. Hence, starting from $c + d\phi$ we apply the first part of the theorem to $\tilde{\psi} = 1/(c+d\phi)$ and construct

It is apparent that (33) and (34) are structurally the same. From Theorem 1 it follows that any further application of Mobius transformations would not lead to anything more general than (34), which itself follows from (29) for the simplifying choice $c = 1$, $d = 0$. Observe that, apart from a constant, u_2 and u in (33) and (34), have the same "singular part" as u in (3). This peculiar result is not unique for the KdV. Even evolution equations such as the Kuramoto-Sivashinsky (KS) equation, which does not pass the P-test in the strict sense (due to the presence of complex conjugate resonances) exhibits the same behavior. Conte and Musette (1988a) have shown similar results for the KS equation, which greatly inspired us to prove theorems 1 and 2 for the famous KdV case.

Example

Let us start from ϕ in (28) and calculate u according to the first three steps outlined in the beginning of this section. The dispersion law follows from (30):

$$\omega = -(\lambda + \delta k^2)k \ . \tag{35}$$

From either (5) or (6) one obtains

$$u_2 = -\frac{1}{\alpha}(\lambda + \frac{3}{2}\delta k^2) . \tag{36}$$

For the present choice of ϕ, the singular part in (3) vanishes, so that $u = u_2$. We now carry out the calculations for

$$\psi = a + be^{\theta} , \tag{37}$$

which results from a special Mobius transformation on e^{θ}, hence λ and ω (see (35)) remain the same. Following the notation of the proof of Theorem 1,

$$v_2 = u = u_2 = -\frac{1}{\alpha}(\lambda + \frac{3}{2}\delta k^2) . \tag{38}$$

Denoting the solution corresponding to ψ by v, we obtain

$$v = \frac{12\delta}{\alpha} (\ell n\psi)_{2x} + v_2$$

$$= \frac{12\delta k^2 abe^{\theta}}{\alpha(a+be^{\theta})^2} - \frac{1}{\alpha}(\lambda + \frac{3}{2}\delta k^2) . \tag{39}$$

This solution depends only on the free parameters λ, k and θ_0, the ratio $|\frac{b}{a}|$ being absorbed in θ_0. For a=0 one obtains the trivial constant solution to the KdV equation. For $\frac{b}{a} > 0$ we retrieve

$$v = \frac{3\delta k^2}{\alpha} \operatorname{sech}^2 \frac{k}{2}[x + (\lambda + \frac{1}{2}\delta k^2)t + \frac{\theta_0}{k}]$$

$$- \frac{1}{\alpha}(\lambda + \frac{3}{2}\delta k^2) \tag{40}$$

with $\frac{b}{a} = \exp \theta_0$.

For $\frac{b}{a} < 0$, the sech^2 in (40) is replaced by $-\operatorname{cosech}^2$, the singular solution to the KdV equation.

Let us conclude this section by pointing out that an equivalent result can be obtained by starting from

$$\psi = A + B \tanh \frac{\theta}{2} , \tag{41}$$

where A and B depend in a suitable way on a,b,c and d in the Mobius group (29), where only the parmeters $\frac{a+b}{b}$ and $\frac{c+d}{d}$ are relevant ($\frac{b}{d}$ can be scaled out). Calculations along similar lines as before produce an elegant, though equivalent, representation of the regular (and singular) traveling wave solution of (2):

$$v = \frac{3\delta k^2}{2\alpha} \left[1 - 2 \left(\frac{B + A \tanh \frac{\theta}{2}}{A + B \tanh \frac{\theta}{2}} \right) \right]^2 - \frac{\lambda}{\alpha} . \tag{42}$$

One easily verifies that (39) and (42) are the same for $A = \frac{1}{2}(a+b)$, $B = \frac{1}{2}(b-a)$. For a graph such that $0 < v < 1$ we selected $\alpha = -2\lambda = 3\delta k^2$ for which (42) reduces to

$$v(\zeta) = \frac{(1-\Delta^2)\,\mathrm{sech}^2\zeta}{(1+\Delta \tanh \zeta)^2} \tag{43}$$

where $\Delta = B/A$, $\zeta = \frac{k}{2}\left(x - \frac{\alpha}{3} t + \frac{\theta_0}{k}\right)$, which is symmetric with respect to $\zeta_0 = -\tanh^{-1}\Delta$. Fig. 1 shows $v(\zeta)$ for $\Delta = \frac{1}{2}$. For any $\Delta > 1$ the solution is singular (cosech^2 - type). At this point one might wonder if it is possible to construct N solitary wave solutions from the PB equations. The answer is no, and this issue will be addressed in the next section.

It is however possible to construct an infinite class of rational solutions to the KdV equation starting from (5) and (6), and by exploiting invariance properties of the related equation (7) other than the Mobius group. Remark that (7) is invariant under the reciprocal derivative transformation $\phi_x = \frac{1}{\psi_x}$ (Weiss 1984a, 1986a,b, 1987). Details on the construction of rational solutions, which is out of the scope of this paper, may be found in papers by Weiss (1984a, 1986b), Gibbon et al (1985) and references therein.

4. Connection with Other Methods

4.1 Hirota's method

Hirota (see, for instance, Ablowitz and Segur 1981, Matsuno 1984) constructs the N solitary wave solution to the KdV equation (2) in a quite ingeneous way. Hirota substitutes

$$u = \frac{12\delta}{\alpha} (\ell n\ f)_{2x} \tag{44}$$

into (2), hence reducing it to a quadratic equation in $f(x,t)$:

$$ff_{xt} - f_x f_t + \delta ff_{4x} - 4\delta f_x f_{3x} + 3\delta f_{2x}^2 = Gf^2, \tag{45}$$

which in turn can be written in bilinear form as

$$(D_x D_t + \delta D_x^4)\, f\bullet f = Gf^2, \tag{46}$$

using the bilinear operator (Newell 1985)

$$D_x f\bullet g = \lim_{y\to 0} \frac{\partial}{\partial y} f(x+y)g(x-y) \tag{47}$$

where y plays the role of an auxiliary variable. Similar formulae hold for D_t and products (powers) of D_x and D_t. Setting the integration constant G in (46) equal to zero, the N solitary wave solution is then obtained as follows. Upon substitution of a formal expansion for f, say

$$f = 1 + \sum_{i=1}^{\infty} \varepsilon^i F^{(i)}(x,t) \tag{48}$$

into (46) and equating the coefficients of the bookkeeping parameter ε order by order, one arrives at a perturbation scheme for the subsequent $F^{(i)}$. Starting from a sum of N real exponentials

$$F^{(1)} = \sum_{j=1}^{N} \exp\,(k_j x - \omega_j t + \theta_j^0) = \sum_{j=1}^{N} e^{\theta_j} \tag{49}$$

the expansion (48) can be broken off exactly at level N (i.e. $F^{(N+1)} = F^{(N+2)} = \ldots = 0$). Calculation of the RHS of (44) gives the explicit form of u(x,t).

As an example, the two solitary wave solution, which we will denote by $u^{(2)}(x,t)$, is obtained by taking N=2 in (49), which will satisfy the first equation in the perturbation scheme provided $\omega_j = \delta k_j^3$ (dispersion law). After some tedious algebra involving the calculation of $F^{(2)}$ and subsequent substitutions of the results in (48) and (44), one gets

$$u^{(2)} = \frac{3\delta}{\alpha}\,(k_2^2 - k_1^2)\,\frac{(k_1^2 \operatorname{sech}^2 \frac{\theta_1}{2} + k_2^2 \operatorname{cosech}^2 \frac{\theta_2}{2})}{(k_1 \tanh \frac{\theta_1}{2} - k_2 \coth \frac{\theta_2}{2})^2} \tag{50}$$

$$= \frac{6\delta}{\alpha}(k_2^2 - k_1^2) \frac{[(k_2^2 - k_1^2) + k_2^2 \cosh\theta_1 + k_1^2 \cosh\theta_2]}{[(k_2-k_1)\cosh(\frac{\theta_1+\theta_2}{2}) + (k_2+k_1)\cosh(\frac{\theta_1-\theta_2}{2})]^2} . \tag{51}$$

Moloney and Hodnett (1986) have shown that the 2 soliton solution can be decomposed into two amplitude and phase modulated solitary waves. For completeness we recall their results here:

$$u^{(2)} = \frac{3\delta}{\alpha}\, [k_1^2 A(\theta_2) \operatorname{sech}^2 \frac{1}{2}(\theta_1 + H(\theta_2))$$

$$+ k_2^2 A(\theta_1) \operatorname{sech}^2 \frac{1}{2}(\theta_2 + H(\theta_1))], \tag{52}$$

with amplitude and phase modulations

$$A(\theta_i) = \frac{1 + B_i e^{\theta_i} + A_{12} e^{2\theta_i}}{(1+e^{\theta_i})(1+A_{12}e^{\theta_i})} \tag{53}$$

$$H(\theta_i) = \ell n\left(\frac{1 + A_{12}e^{\theta_i}}{1 + e^{\theta_i}}\right) \tag{54}$$

with

$$A_{12} = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 \tag{55}$$

and

$$k_1^2 B_2 + k_2^2 B_1 = 2(k_1 - k_2)^2 . \tag{56}$$

The ambiguity in selecting B_1, B_2 can be resolved by requiring that the total area under each of the pulses in (52) is preserved and constant. Mathematically this implies that

$$B_1 = -B_2 = -2\left(\frac{k_1-k_2}{k_1+k_2}\right) , \tag{57}$$

for which clearly (56) is satisfied. Motivated by the similarity in the starting points (44) of Hirota's method and (3) for the Painleve approach, one may search for underlying connections. More specifically, one may wonder whether or not it is possible to construct multisoliton solutions from Painleve-Backlund equations (5) and (6). Recall that these were derived by setting the terms in negative powers of ϕ independently equal to zero, whereas for (45) there was no such restriction. Hence, multiplication of (5) with ϕ and subtraction of (6) yields

$$\begin{aligned}
&\phi\phi_{xt} - \phi_x\phi_t + \delta\phi\phi_{4x} - 4\delta\phi_x\phi_{3x} + 3\delta\phi_{2x}^2 \\
&\quad = -\alpha u_2\phi^2(\ell n\phi)_{2x} \\
&\quad = -\alpha u_2(\phi_{2x}\phi - \phi_x^2)\ . \qquad (58)
\end{aligned}$$

We can now see that Painleve-Backlund equations imply (45) for G=0 provided $u_2 = 0$ and ϕ is exponential in nature. For $\phi = a + b\exp(kx - \omega t + \theta_0)$ (58) is satisfied since (5) and (6) are with

$$u_2 = \frac{1}{\alpha}\left(\frac{\omega}{k} - \delta k^2\right). \qquad (59)$$

For $u_2 = 0$ we recover the dispersion law $\omega = \delta k^3$ used in Hirota's method and f and ϕ coincide. In Hirota's formalism a two soliton solution is obtained with

$$f = 1 + e^{\theta_1} + e^{\theta_2} + \left(\frac{k_1-k_2}{k_1+k_2}\right) e^{\theta_1 + \theta_2} \qquad (60)$$

which surely satisfies (45) for G=0, but could never fulfill (5) and (6) simultaneously for u_2. Hence it appears impossible to construct multi-solitary wave solutions from the PB equations for a truncated Laurent expansion. We strongly believe that even the complete Laurent series (see Weiss et al 1983) would fail to reconstruct (50).

As a final remark, it is possible to find simple solutions of (45) for $G \neq 0$. For instance, recalling that two functions f which differ by the exponential $\exp(\alpha(t)x + \beta(t))$, with arbitrary $\alpha(t)$ and $\beta(t)$, will give rise to the same u through (44),

$$f = \cosh\left(\frac{\theta}{2}\right) \exp(-M\theta^2), \tag{61}$$

with

$$\theta = k(x + \lambda t + \frac{1}{2}\delta k^2 t) + \theta_0,$$

and

$$M = -\frac{(\lambda + \frac{3}{2}\delta k^2)}{24\delta k^2}$$

satisfies (45) provided

$$G = -(2\lambda - \delta k^2)(2\lambda + 3\delta k^2). \tag{62}$$

The corresponding solitary wave solution is in (40), from which f in (61) was obtained by two integrations according to the RHS of (44).

4.2 Lax method and Inverse Scattering Transform (IST)

The Lax pair for the KdV equations (Lax 1968) can be derived from the PB equations (5), (6) and (7) in an easy way. First, we solve (6) for ϕ_t and substitute this into (7), yielding

$$3\delta\left(\frac{\phi_{3x}}{\phi_x} - \frac{1}{2}\frac{\phi_{2x}^2}{\phi_x^2}\right) + (\alpha u_2 + \lambda) = 0. \tag{63}$$

Secondly, differentiation of ϕ_t with respect to x allows to eliminate ϕ_{xt} between (5) and (6) resulting in

$$\alpha\phi_x^3(u_2)_x + 3\delta\phi_x^2\phi_{4x} - 6\delta\phi_x\phi_{2x}\phi_{3x} + 3\delta\phi_{2x}^3 = 0. \tag{64}$$

Next, following Weiss et al (1983), we substitute

$$\phi_x = V^2 \tag{65}$$

into (63), to get the Schrodinger equation (first Lax equation)

$$6\delta V_{2x} + (\alpha u_2 + \lambda)V = 0. \tag{66}$$

Substitution of (65) into (5) and (64) and elimination of the nonlinear term $V_x V_{2x}$ between the resulting equations gives the second Lax equation

$$V_t + \alpha u_2 V_x + \frac{\alpha}{2}(u_2)_x V + 4\delta V_{3x} = 0. \tag{67}$$

Defining the Lax operators

$$L = \frac{6\delta\beta}{\alpha}\ \frac{\partial^2 \bullet}{\partial x^2} + \beta u_2\ ,\ \beta \in \mathbb{R}, \tag{68}$$

$$B = -\left[4\delta\ \frac{\partial \bullet}{\partial x^3} + \frac{\alpha}{2}\ (u_2\ \frac{\partial \bullet}{\partial x} + \frac{\partial}{\partial x}\ \ u_2)\right], \tag{69}$$

we recast (66) and (67) into a compact form $LV = \Lambda V$ with $\Lambda = -\lambda/\alpha$, and $V_t = BV$. One can easily verify that the KdV equation (2) in u_2 is equivalent to

$$L_t = [B,L] = BL - LB. \tag{70}$$

This result was first obtained by Weiss et al (1983), and Weiss (1983) in two different ways, and the same procedures for obtaining Lax pairs apply for a large variety of ODEs, systems of ODEs and PDEs. For the latest results we refer to Newell et al (1987). From (65) the connection between the Painleve function ϕ and the eigenfunction V in the Schrodinger equation is clear.

4.3 The Backlund transformation method

Once the Lax pair is available, it is rather straightforward to obtain the Miura transformation and even the traditional Backlund transformation. Indeed, if we introduce a new dependent variable $r(x,t)$ by the Cole-Hopf transformation (Newell 1985)

$$r(x,t) = \frac{V_x}{V} = \frac{\partial}{\partial x} (\ell n\ V) \tag{71}$$

or, equivalently.

$$V = \exp \int^x r(s,t)ds, \tag{72}$$

then the Schrodinger equation may be replaced by

$$\frac{6\delta\beta}{\alpha}(r_x + r^2) + \beta u_2 = \Lambda, \tag{73}$$

which is known as a Miura transformation between the functions r and u_2. Since u_2 satisfies the KdV it is reasonable to expect that r will satisfy the modified KdV (mKdV). To show this we first rewrite the second Lax equation entirely in terms of r and u_2:

$$\begin{aligned} -r_t &= 12\delta r^2 r_x + 12\delta r_x{}^2 + 12\delta r r_{2x} \\ &\quad + 4\delta r_{3x} + \alpha(u_2)_x r + \alpha(u_2) r_x + \frac{\alpha}{2}(u_2)_{2x}. \end{aligned} \tag{74}$$

Subsequently, we eliminate u_2 between (73) and (74), yielding

$$r_t - 6\delta r^2 r_x + \frac{\alpha\Lambda}{\beta} r_x + \delta r_{3x} = 0, \tag{75}$$

which indeed is the mKdV. The traditional Backlund transformation now follows from a simple, but clever argument. Observe that if r is a solution to (75), (corresponding to u_2) then $-r$ is a solution as well (corresponding to $\tilde{u}_2$, another solution to the KdV),

and

$$\frac{6\delta\beta}{\alpha}(-r_x + r^2) + \beta\tilde{u}_2 = \Lambda. \tag{76}$$

Subtracting (76) from (73) we obtain

$$r = \frac{\alpha}{12\delta}(\tilde{q} - q). \tag{77}$$

where $q_x = u_2$, $\tilde{q}_x = \tilde{u}_2$, q and $\tilde{q}$ being the potentials. Substituting the last result back into (73), we get the classical Backlund transformation,

$$(\tilde{q} + q)_x = \frac{2\Lambda}{\beta} - \frac{\alpha}{12\delta}(\tilde{q} - q)^2 . \tag{78}$$

A similar, but more complicated relation for $(\tilde{q} - q)_t$, follows from (74), or a slightly more symmetric form (75). Wahlquist and Estabrook (1973) have shown that this evolution equation together with (78) then form a completely integrable Phaffian system. Integrability being assured, it suffices to only consider (78). Combining (71) and (77) we also have

$$\tilde{q} - q = \frac{12\delta}{\alpha}\frac{\partial}{\partial x}(\ell n\ V) \tag{79}$$

or

$$\tilde{u}_2 - u_2 = \frac{12\delta}{\alpha}\frac{\partial^2}{\partial x^2}(\ell n\ V) = \frac{6\delta}{\alpha}\frac{\phi_{2x}}{\phi_x}, \tag{80}$$

using (65).

Keep in mind here that u_2 and $\tilde{u}_2$ are not arbitrary and independent solutions to the KdV. To be precise, u_2 is related to r and $\tilde{u}_2$ to $-r$. From (71) it then follows that u_2 corresponds to V whereas $\tilde{u}_2$ corresponds to $1/V$. Going back to (65), u_2 corresponds to ϕ whereas $\tilde{u}_2$ corresponds to Γ defined as

$$\Gamma_x = \frac{1}{\phi_x} . \tag{81}$$

The Backlund transformation (80), which involves $E = \phi_{2x}/\phi_x$ as defined in (11), suggests that (7), which only involves ϕ and its partial derivatives, is invariant for the change $\phi \to \Gamma$. This result, which was first observed by Weiss (1984a), is easy to verify by direct calculation. Using (81) in combination with (80) gives

$$\tilde{u}_2 - u_2 = -\frac{6\delta}{\alpha}\frac{\Gamma_{2x}}{\Gamma_x} . \tag{82}$$

Furthermore, under the change of the dependent variable in (81),

$$E_\phi = \frac{\phi_{2x}}{\phi_x} = -E_\Gamma = -\frac{\Gamma_{2x}}{\Gamma_x} , \tag{83}$$

$$C_\phi = \frac{\phi_t}{\phi_x} = C_\Gamma + 2\delta(E_\Gamma)_x , \tag{84}$$

$$S_\phi = (E_\phi)_x - \frac{1}{2}(E_\phi)^2 = S_\Gamma - 2(E_\Gamma)_x , \tag{85}$$

such that (7) remains invariant. The solution corresponding to ϕ, denoted by $u_2 = (u_2)_\phi$ will be transformed into $\tilde{u}_2 = (u_2)_\Gamma$ according to (80) or (82). Remark that in eqs. (83)-(85) and the discussion above the subscripts ϕ and Γ do not represent partial derivatives. For a further discussion of the Backlund transformation (78) we refer to Wahlquist and Estabrook (1973).

5. Conclusion

Painleve analysis of the KdV leads to a non-standard Backlund transformation involving a (singular) manifold ϕ which has to satisfy simultaneously two partial differential equations in which the other solution u_2 to the KdV occurs as a coefficient. Upon elimination of the latter (i.e. u_2), one obtains a partial differential in ϕ which has two terms, the ratio ϕ_t/ϕ_x and the Schwarzian in ϕ, that are invariant under the Mobius group and a reciprocal derivative transformation. These invariances allow us to construct a hierarchy of new solutions, at least if the initial ϕ is rational. For ϕ of exponential form it is impossible to go higher than one rung on the ladder: the Mobius group remains invariant up to a trivial change of constants. The meaning of the function ϕ is illuminated through the connections of the Painleve analysis with Hirota's perturbation method, the Lax method, the Inverse Scattering Technique and the standard Backlund tansformation method.

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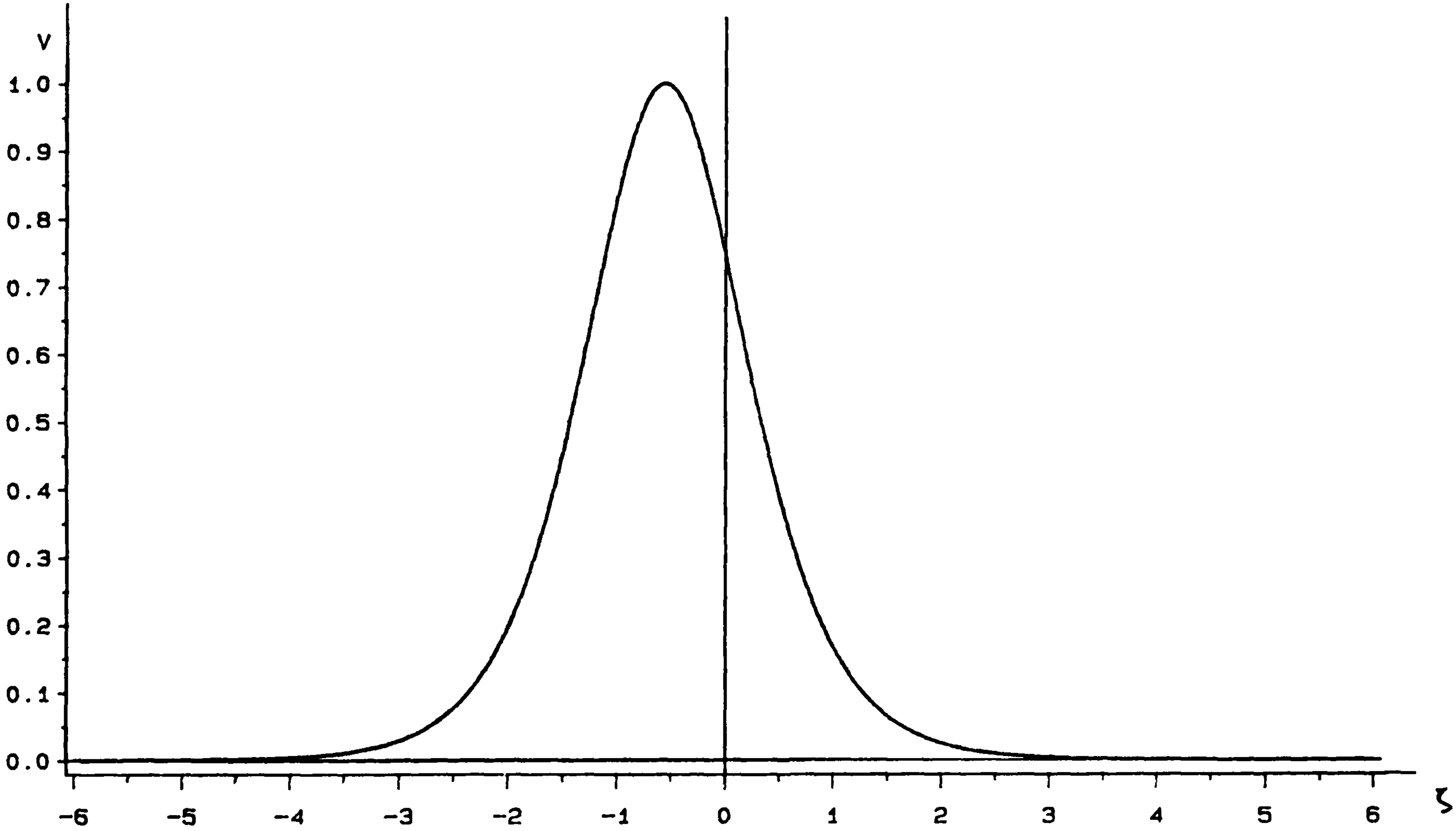

FIG.1 SINGLE SOLITARY WAVE SOLUTION
FOR C=1/2